FINITE RANK OPERATORS IN LIE IDEALS OF NEST ALGEBRAS

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Abstract. The main theorem provides a characterisation of the finite rank operators lying in a norm closed Lie ideal of a continuous nest algebra. These operators are charaterised as those finite rank operators in the nest algebra satisfying a condition determined by a left order continuous homomorphism on the nest. A crucial fact used in the proof of this theorem is the decomposability of the finite rank operators. One shows that a finite rank operator in a norm closed Lie ideal of a continuous nest algebra can be written as a finite sum of rank one operators lying in the ideal.

1. Introduction

The structure of Lie ideals of nest algebras has been investigated by many authors for at least a decade (cf. [3, 8, 9, 10, 11] and the literature referenced therein). Two main lines of this research might be essentially described as focusing either on the connection between Lie ideals and associative ideals or on similarity invariant subspaces.

The present work approaches the investigation of Lie ideals from a different perspective, its departing point being the assumption that the set of finite rank operators in the ideal should be worth investigating. This seems only natural if one bears in mind the important rôle played by the finite rank operators in the theory of nest algebras given their density properties (cf. [2, 6]). Furthermore, investigating the decomposability of the finite rank operators in Lie ideals seems to be an interesting problem in its own right. One has simply to refer to the literature to realise that an effort has been made to investigate the decomposability of finite rank operators lying in various subalgebras of $\mathcal{B}(\mathfrak{H})$ (see, for example, [1, 4, 5, 7] and also [6], citing a theorem of Ringrose).

The main result of this work, Theorem 4.3, appears in Section 4. This theorem provides a characterisation of the finite rank operators lying in a norm closed Lie ideal $\mathfrak{L}$ of a continuous nest algebra $\mathcal{T}(\mathcal{N})$. This is a characterisation along the lines of that obtained by Erdos and Power for weakly closed associative ideals (cf. [7], Theorem 1.5) and characterises the finite rank operators lying in $\mathfrak{L}$ as those in the nest algebra satisfying a

1991 *Mathematics Subject Classification.* Primary 47L35; Secondary 17B60.

Key words and phrases. Finite rank operator, Lie ideal, nest algebra, rank one operator.

condition determined by a left order continuous homomorphism on the nest $\mathcal{N}$.

A crucial fact used in the proof of Theorem 4.3 is that a finite rank operator in a norm closed Lie ideal of a continuous nest algebra can be written as a finite sum of rank one operators lying in the ideal. The decomposability of the finite rank operators is stated in Theorem 3.5, which is the principal result of Section 3. The remaining contents of this section consists in the statement and proof of auxiliary results required to prove Theorem 3.5.

Of the four sections of which this work consists, only Section 2 remains to be described. This is a preliminary section establishing notation and recalling well-known facts about nest algebras needed in the sequel.

2. Preliminaries

Let $\mathfrak{H}$ be a complex Hilbert space and let $\mathcal{B}(\mathfrak{H})$ be the complex Banach algebra of bounded linear operators on $\mathfrak{H}$. A totally ordered family $\mathcal{N}$ of projections in $\mathcal{B}(\mathfrak{H})$ containing 0 and the identity I is said to be a *nest*. If, furthermore, $\mathcal{N}$ is a complete sublattice of the lattice of projections in $\mathcal{B}(\mathfrak{H})$, then $\mathcal{N}$ is called a *complete nest*.

Let P be a projection in the nest $\mathcal{N}$ and define P_- to be the projection in $\mathcal{N}$ satisfying

$$P_- = \bigvee\{Q \in \mathcal{N} : Q < P\},$$

if P is not zero, and P_- is zero, otherwise. A complete nest $\mathcal{N}$ is said to be *continuous* if, for all projections P in $\mathcal{N}$, the projections P and P_- coincide.

The *nest algebra* $\mathcal{T}(\mathcal{N})$ associated with a nest $\mathcal{N}$ is the subalgebra of all operators T in $\mathcal{B}(\mathfrak{H})$ such that, for all projections P in $\mathcal{N}$,

$$T(P(\mathfrak{H})) \subseteq P(\mathfrak{H}),$$

or, equivalently, an operator T in $\mathcal{B}(\mathfrak{H})$ lies in $\mathcal{T}(\mathcal{N})$ if and only if, for all projections P in the nest $\mathcal{N}$,

$$P^{\perp}TP = 0.$$

It is well-known that $\mathcal{T}(\mathcal{N})$ is a unital weak operator closed subalgebra of $\mathcal{B}(\mathfrak{H})$ and that each nest is contained in a complete nest which generates the same nest algebra (cf. [2, 13]). Since this work is mainly concerned with the nest algebras and not with the nests themselves, henceforth only complete nests will be considered. A nest algebra associated to a continuous nest is said to be a *continuous* nest algebra.

A nest algebra $\mathcal{T}(\mathcal{N})$ together with the product, defined for all operators T and S in $\mathcal{T}(\mathcal{N})$, by

$$[T, S] = TS - ST$$

is a Lie algebra and a complex subspace $\mathfrak{L}$ of the nest algebra $\mathcal{T}(\mathcal{N})$ is said to be a *Lie ideal* if $[\mathfrak{L}, \mathcal{T}(\mathcal{N})] \subseteq \mathfrak{L}$.

Let x and y be elements of the Hilbert space $\mathfrak{H}$ and let $x \otimes y$ be the rank one operator defined, for all z in $\mathfrak{H}$, by

$$z \mapsto \langle z, x \rangle y,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product of $\mathfrak{H}$. A rank one operator $x \otimes y$ lies in $\mathcal{T}(\mathcal{N})$ if and only if

$$P_{-}x = 0 \qquad \text{and} \qquad Py = y,$$

for some projection P in the nest, and P can be chosen to be equal to $\bigwedge\{Q \in \mathcal{N} : Qy = y\}$ (cf. [13]). As a consequence, since the nest $\mathcal{N}$ is continuous, for any $x \otimes y$ in $\mathcal{T}(\mathcal{N})$, the elements x and y are necessarily orthogonal. (For the general theory of nest algebras, the reader is referred to [2, 13].)

3. Finite rank operators

The principal result of this section, Theorem 3.5, establishes the possibility of constructing the finite rank operators in $\mathfrak{L}$ as finite sums of rank one operators also lying in the ideal. A central idea underlies the proof of this theorem, and implicitly also the proofs of some of the auxiliary results. It might be described as regarding each rank one operator $x \otimes y$ in any nest algebra $\mathcal{T}(\mathcal{N})$ from the point of view of a particular pair of projections intrinsically associated to x and y. The way in which this projections are built is described as follows.

Let z be an element of the Hilbert space $\mathfrak{H}$ and, as in [12], let the projections P_z and $\hat{P}_z$ be defined by

$$P_z = \bigwedge\{Q \in \mathcal{N} : Qz = z\} \quad , \qquad \hat{P}_z = \bigvee\{Q \in \mathcal{N} : Qz = 0\}.$$

The projections P_z and $\hat{P}_z$ lie in the nest $\mathcal{N}$ and are such that

$$P_z z = z\,, \qquad\qquad \hat{P}_z z = 0.$$

The pair of projections associated to the rank one operator $x \otimes y$ are $\hat{P}_x$ and P_y, respectively. The proof of the next lemma is a first example of how these projections are used. In the lemma, for a given norm closed Lie ideal $\mathfrak{L}$ of a continuous nest algebra, some rank one operators are singled out as lying necessarily in $\mathfrak{L}$, provided that a particular operator $x \otimes y$ lies in $\mathfrak{L}$.

Lemma 3.1. *Let $\mathcal{T}(\mathcal{N})$ be a continuous nest algebra, let $\mathfrak{L}$ be a norm closed Lie ideal of $\mathcal{T}(\mathcal{N})$ and let $x \otimes y$ be a rank one operator lying in $\mathfrak{L}$. The following assertions hold.*

(i) *If $x \otimes z$ is a rank one operator in the nest algebra $\mathcal{T}(\mathcal{N})$ such that $P_z \leq P_y$, then $x \otimes z$ lie in $\mathfrak{L}$.*
(ii) *If $w \otimes y$ is a rank one operator in the nest algebra $\mathcal{T}(\mathcal{N})$ such that $\hat{P}_x \leq \hat{P}_w$, then $w \otimes y$ lies in $\mathfrak{L}$.*

Proof. To prove assertion (i), firstly, suppose that $P_z < P_y$ and let y' be the non zero element of $\mathfrak{H}$ defined by

$$y' = P_z^{\perp} y.$$

Since $y' \otimes z$ lies in $\mathcal{T}(\mathcal{N})$, the bracket

$$\begin{aligned}[x \otimes y, y' \otimes z] &= < z, x > (y' \otimes y) - < y, y' > (x \otimes z) \\ &= - < y, y' > (x \otimes z) = -\|y'\|^2 (x \otimes z)\end{aligned}$$

lies in $\mathfrak{L}$, yielding that also $x \otimes z$ lies in $\mathfrak{L}$.

If $P_z = P_y$, then

$$P_y(\mathfrak{H}) = P_z(\mathfrak{H}) = \overline{\bigcup_{P \in \mathcal{N}, P < P_z} P(\mathfrak{H})},$$

and, consequently, there exists a sequence (z_n) in

$$\bigcup_{P \in \mathcal{N}, P < P_z} P(\mathfrak{H})$$

converging to z in the norm topology. It follows that $(x \otimes z_n)$ is a convergent sequence in $\mathfrak{L}$ whose limit $x \otimes z$ also lies in $\mathfrak{L}$.

To prove assertion (ii), suppose initially that $\hat{P}_x < \hat{P}_w$. Let z be the element of the Hilbert space $\mathfrak{H}$ defined by

$$z = \hat{P}_w x.$$

The operator

$$\begin{aligned}[x \otimes y, w \otimes z] &= < z, x > (w \otimes y) - < y, w > (x \otimes z) \\ &= \|\hat{P}_w x\|^2 (w \otimes y),\end{aligned}$$

lies in $\mathfrak{L}$ and, therefore, also $w \otimes y$ lies in $\mathfrak{L}$.

Suppose now that $\hat{P}_x = \hat{P}_w$. Let (w_n) be a sequence in

$$\bigcup_{P \in \mathcal{N}, \hat{P}_x < P} P^{\perp}(\mathfrak{H})$$

converging to w in the norm topology. Since, for all n, the operator $w_n \otimes y$ lies in $\mathfrak{L}$ and the sequence $(w_n \otimes y)$ converges to $w \otimes y$ in the norm topology, it follows that $w \otimes y$ lies in $\mathfrak{L}$. □

It is now possible to identify a whole "corner" of rank one operators in $\mathcal{T}(\mathcal{N})$ as also lying in the Lie ideal $\mathfrak{L}$.

Theorem 3.2. *Let $\mathcal{T}(\mathcal{N})$ be a continuous nest algebra, let $\mathfrak{L}$ be a norm closed Lie ideal of $\mathcal{T}(\mathcal{N})$, let $x \otimes y$ be a rank one operator lying in $\mathfrak{L}$ and let $w \otimes z$ be a rank one operator in the nest algebra $\mathcal{T}(\mathcal{N})$ satisfying*

$$\hat{P}_x \leq \hat{P}_w \qquad \text{and} \qquad P_z \leq P_y.$$

Then, the operator $w \otimes z$ lies in $\mathfrak{L}$.

Proof. By Lemma 3.1 (ii), the operator $x \otimes z$ lies in $\mathfrak{L}$ and applying (i) of the same lemma to this operator, it follows that $w \otimes z$ lies in $\mathfrak{L}$. □

Lemma 3.3. *Let $\mathcal{T}(\mathcal{N})$ be a continuous nest algebra, let $\mathfrak{L}$ be a norm closed Lie ideal of $\mathcal{T}(\mathcal{N})$ and let*

$$T = \sum_{i=1}^{n} x_i \otimes y_i,$$

be a rank n operator lying in $\mathfrak{L}$, with $n \geq 1$, where, for all $i, j = 1, \ldots, n$, the operator $x_i \otimes y_i$ lies in $\mathcal{T}(\mathcal{N})$ and $\hat{P}_{x_i} < \hat{P}_{x_j}$, whenever $i < j$. Then, for all $i = 1, \ldots, n$, the rank one operator $x_i \otimes y_i$ lies in $\mathfrak{L}$.

Proof. The assertion trivially holds for $n = 1$. Suppose now that n is greater than 1. The operator

$$\begin{aligned}
\left[\hat{P}_{x_n}, T\right] &= \hat{P}_{x_n}\Big(\sum_{i=1}^{n} x_i \otimes y_i\Big) - \Big(\sum_{i=1}^{n} x_i \otimes y_i\Big)\hat{P}_{x_n} \\
&= \sum_{i=1}^{n} x_i \otimes (\hat{P}_{x_n} y_i) - \sum_{i=1}^{n} (\hat{P}_{x_n} x_i) \otimes y_i \\
&= \sum_{i=1}^{n} x_i \otimes y_i - \sum_{i=1}^{n-1} (\hat{P}_{x_n} x_i) \otimes y_i \\
&= T - \sum_{i=1}^{n-1} (\hat{P}_{x_n} x_i) \otimes y_i
\end{aligned}$$

lies in $\mathfrak{L}$. If n coincides with 2, then $(\hat{P}_{x_2} x_1) \otimes y_1$ lies in $\mathfrak{L}$ and, by Theorem 3.2, the operator $x_1 \otimes y_1$ lies in $\mathfrak{L}$. Hence, $x_2 \otimes y_2$ also lies in $\mathfrak{L}$ and thus the proof is complete. In the case of n being greater than 2, observe that the subset $\mathfrak{X}$ of the Hilbert space $\mathfrak{H}$ defined by

$$\mathfrak{X} = \{\hat{P}_{x_n} x_i : i = 1, \ldots, n-1\}$$

is linearly independent. In fact, if $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ are scalars such that

$$\alpha_1 \hat{P}_{x_n} x_1 + \alpha_2 \hat{P}_{x_n} x_2 + \cdots + \alpha_{n-1} \hat{P}_{x_n} x_{n-1} = 0,$$

then, since

$$\hat{P}_{x_1} < \hat{P}_{x_2} < \cdots < \hat{P}_{x_{n-1}} < \hat{P}_{x_n},$$

it follows that

$$\begin{aligned}
\hat{P}_{x_2}(\alpha_1 \hat{P}_{x_n} x_1 + \alpha_2 \hat{P}_{x_n} x_2 + \cdots + \alpha_{n-1} \hat{P}_{x_n} x_{n-1}) &= \alpha_1 \hat{P}_{x_2} x_1 \\
&= 0.
\end{aligned}$$

Consequently, α_1 must be equal to zero since $\hat{P}_{x_2} x_1$ is different from zero. Similarly, using now the projection $\hat{P}_{x_3}$ and the equality

$$\alpha_2 \hat{P}_{x_n} x_2 + \cdots + \alpha_{n-1} \hat{P}_{x_n} x_{n-1} = 0,$$

one has that α_2 must coincide with zero. It is clear that a repetition of this reasoning yields that, for all i in the set $\{1,\dots,n-1\}$, the scalar α_i must coincide with zero and, therefore, the set $\mathfrak{X}$ is linearly independent. Moreover, since the set $\{y_i : i=1,\dots,n\}$ is linearly independent, and thus also the set $\{y_i : i=1,\dots,n\}$ is linearly independent, it follows that the operator

$$T_1 = \sum_{i=1}^{n-1} (\hat{P}_{x_n} x_i) \otimes y_i$$

has rank $n-1$ and lies in $\mathfrak{L}$.

Analogously, one obtains the equality

$$\left[\hat{P}_{x_{n-1}}, T_1\right] = T_1 - \sum_{i=1}^{n-2} (\hat{P}_{x_{n-1}} x_i) \otimes y_i$$

for the operator T_1 and a similar reasoning yields that the rank $n-2$ operator

$$T_2 = \sum_{i=1}^{n-2} (\hat{P}_{x_{n-1}} x_i) \otimes y_i$$

lies in $\mathfrak{L}$. Repeating this procedure as many times as required, one has that

$$T_{n-1} = (\hat{P}_{x_2} x_1) \otimes y_1$$

lies in $\mathfrak{L}$. Hence, by Theorem 3.2 and observing that, for all $i=2,\dots,n$,

$$\hat{P}_{\hat{P}_{x_i} x_1} = \hat{P}_{x_1},$$

it follows that the operator $x_1 \otimes y_1$ and also all the operators $(\hat{P}_{x_i})x_1 \otimes y_1$ lie in the ideal $\mathfrak{L}$.

Back substitution in the equality

$$T_{n-2} = (\hat{P}_{x_3} x_1) \otimes y_1 + (\hat{P}_{x_3} x_2) \otimes y_2,$$

similarly yields that, for all $i=3,\dots,n$, the operator $(\hat{P}_{x_3} x_2) \otimes y_2$ and also the operator $x_2 \otimes y_2$ lie in $\mathfrak{L}$. Clearly, the proof is complete after repeating this reasoning sufficiently many times. $\square$

Lemma 3.4. *Let $\mathcal{T}(\mathcal{N})$ be a continuous nest algebra, let $\mathfrak{L}$ be a norm closed Lie ideal of $\mathcal{T}(\mathcal{N})$ and let*

$$T = \sum_{i=1}^{n} x_i \otimes y_i,$$

be a rank n operator lying in $\mathfrak{L}$, with $n \geq 1$, where, for all $i,j=1,\dots,n$, the operator $x_i \otimes y_i$ lies in $\mathcal{T}(\mathcal{N})$ and $\hat{P}_{x_i} = \hat{P}_{x_j}$. Then, T can be written as a finite sum of rank one operators lying in $\mathfrak{L}$.

Proof. The proof will be carried out for $n > 1$, since the result trivially holds for $n = 1$.

Define, for all $P \in \mathcal{N}$, the set $\mathfrak{X}_P$ by

$$\mathfrak{X}_P = \{Px_1, Px_2, \dots, Px_n\},$$

and define the projection Q by

$$Q = \bigwedge \{P \in \mathcal{N} : \mathfrak{X}_P \text{ is linearly independent}\}.$$

Observe that, since the operator T has rank n, the sets $\{x_i : i = 1, \dots, n\}$ and $\{y_i : i = 1, \dots, n\}$ are linearly independent.

Depending on the situation, it may happen that either $\hat{P}_{x_1} = Q$ or $\hat{P}_{x_1} < Q$. This proof is accordingly divided in the two parts a) and b).

a) If $\hat{P}_{x_1} = Q$, then there exists a decreasing net (Q_j) of projections in the nest that converges to $\hat{P}_{x_1}$ in the order topology and such that, for all j, the set $\{Q_j x_i : i = 1, \dots, n\}$ is linearly independent.

Let k be a positive integer lying in the set $\{i = 1, \dots, n\}$, let

$$\mathfrak{S}_k = \operatorname{span}\{Q_j x_i : i = 1, \dots, k-1, k+1, \dots, n\}$$

and let

$$Q_j(\mathfrak{H}) = \mathfrak{S}_k \oplus {\mathfrak{S}_k}^{\perp}$$

be the decomposition of the Hilbert space $Q_j(\mathfrak{H})$ into the direct sum of $\mathfrak{S}$ and its orthogonal complement $\mathfrak{S}^{\perp}$ in the space $Q_j(\mathfrak{H})$. Let

$$Q_j x_k = (Q_j x_k)_p + (Q_j x_k)_o$$

be the orthogonal decomposition of $Q_j x_k$ in $Q_j(\mathfrak{H})$ relatively to the direct sum above. Let w_j and z_j be the elements of the Hilbert space $\mathfrak{H}$ defined by

$$w_j = Q_j^{\perp} x_k \ , \qquad z_j = \frac{1}{\|(Q_j x_k)_o\|^2} (Q_j x_k)_o$$

and consider the operator $w_j \otimes z_j$. Clearly, the operator $w_j \otimes z_j$ lies in the nest algebra $\mathcal{T}(\mathcal{N})$ and, therefore, the operator

$$\begin{aligned}
[T, w_j \otimes z_j] &= \sum_{i=1}^{n} \langle z_j, x_i \rangle (w_j \otimes y_i) - \sum_{i=1}^{n} \langle y_i, w_j \rangle (x_i \otimes z_j) \\
&= \sum_{i=1}^{n} \frac{1}{\|(Q_j x_k)_o\|^2} \langle (Q_j x_k)_o, x_i \rangle \big((Q_j^{\perp} x_k) \otimes y_i \big) \\
&= \sum_{i=1}^{n} \frac{1}{\|(Q_j x_k)_o\|^2} \langle (Q_j x_k)_o, Q_j x_i \rangle \big((Q_j^{\perp} x_k) \otimes y_i \big) \\
&= \frac{1}{\|(Q_j x_k)_o\|^2} \langle (Q_j x_k)_o, Q_j x_k \rangle \big((Q_j^{\perp} x_k) \otimes y_k \big) \\
&= \frac{1}{\|(Q_j x_k)_o\|^2} \langle (Q_j x_k)_o, (Q_j x_k)_o \rangle \big((Q_j^{\perp} x_k) \otimes y_k \big) \\
&= (Q_j^{\perp} x_k) \otimes y_k
\end{aligned}$$

lies in $\mathfrak{L}$. Since the net (Q_j) converges to $\hat{P}_{x_1}$ in the order topology, by [6], Lemma 1, the net $(Q_j x_k)$ converges $\hat{P}_{x_1} x_k$ in the norm topology. Observing that $\hat{P}_{x_1} x_k$ coincides with zero, it follows that the net $(Q_j^{\perp} x_k)$ converges to x_k and, hence,

$$(Q_j^{\perp} x_k) \otimes y_k \longrightarrow x_k \otimes y_k.$$

Since the Lie ideal $\mathfrak{L}$ is norm closed, one has that the operator $x_k \otimes y_k$ lies in $\mathfrak{L}$, thus ending the proof of part a).

b) Suppose now that the projection $\hat{P}_{x_1}$ is less than the projection Q. Let (P_j) be an increasing net converging to Q in the order topology and let $\mathfrak{X}_j$ be the set defined, for all j, by

$$\mathfrak{X}_j = \{P_j x_i : i = 1, \dots, n\}.$$

By the definition of the projection Q, all the sets $\mathfrak{X}_j$ are linearly dependent or, equivalently, for all j, the Grammian determinant Δ_j, defined by

$$\Delta_j = \det \big[\langle P_j x_i, P_j x_k \rangle \big]_{i,k=1,\dots,n}$$

coincides with zero. Since, by [6], Lemma 1, for all $i = 1, \dots, n$, the net $(P_j x_i)$ converges to $Q x_i$ in the norm topology, the continuity of the determinant function yields that the net (Δ_j) converges to the Grammian determinant Δ defined by

$$\Delta = \det \big[\langle Q x_i, Q x_k \rangle \big]_{i,k=1,\dots,n}.$$

Therefore, the determinant Δ coincides with zero and the set $\{Q x_i : i = 1, \dots, n\}$ is linearly dependent.

It is now clear that the projection Q cannot be equal to the identity I. In fact, if Q were equal to I, then the set $\{x_i : i = 1, \dots, n\}$ would be a linearly dependent set, yielding a contradiction.

Let (Q_j) be a decreasing net of projections in the nest converging to Q in the order topology and such that, for all j, the set $\{Q_j x_i : i = 1, \dots, n\}$ is linearly independent.

Let k be a positive integer lying in the set $\{i = 1, \dots, n\}$ and, similarly to part a) of this proof, let

$$\mathfrak{S}_k = \operatorname{span}\{Q_j x_i : i = 1, \dots, k-1, k+1, \dots, n\},$$

let

$$Q_j(\mathfrak{H}) = \mathfrak{S}_k \oplus {\mathfrak{S}_k}^{\perp}$$

and let

$$Q_j x_k = (Q_j x_k)_p + (Q_j x_k)_o$$

be the orthogonal decomposition of $Q_j x_k$ in $Q_j(\mathfrak{H})$ relatively to the direct sum above. Let w_j and z_j be the elements of the Hilbert space $\mathfrak{H}$ defined by

$$w_j = Q_j^{\perp} x_k \ , \qquad\qquad z_j = \frac{1}{\|(Q_j x_k)_o\|^2}(Q_j x_k)_o$$

and consider the operator $w_j \otimes z_j$ in the nest algebra $\mathcal{T}(\mathcal{N})$. Similar computations to those of part a), yield that the operator $[T, w_j \otimes z_j]$ lies in the Lie ideal $\mathfrak{L}$ and satisfies the equality

$$[T, w_j \otimes z_j] = (Q_j^{\perp} x_k) \otimes y_k.$$

Since the net $(Q_j^{\perp})$ converges to $Q^{\perp}$ in the order topology, by [6], Lemma 1, the net $(Q_j^{\perp} x_k)$ converges $Q^{\perp} x_k$ in the norm topology. Hence, it follows that, for all k in the set $\{1, \dots, n\}$, the operator $(Q^{\perp} x_k) \otimes y_k$ lies in $\mathfrak{L}$, yielding that the operator $TQ^{\perp}$ can be written as a finite sum of rank one operators lying in $\mathfrak{L}$. Consequently, $TQ^{\perp}$ and also TQ lie in $\mathfrak{L}$.

Let n_1 be the rank of TQ, and observe that $n_1 < n$ since the set $\{Qx_i : i = 1, 2, \dots, n\}$ is linearly dependent. In fact,

$$\begin{aligned} TQ &= \sum_{i=1}^{n} Qx_i \otimes y_i \\ &= \sum_{r=1}^{n_1} x_{i_r} \otimes y'_{i_r}, \end{aligned}$$

where the mapping $r \mapsto i_r$, from $\{1, \dots, n_1\}$ into $\{1, \dots, n\}$ is injective and, for all indices r, the set $\{y'_{i_r} : r = 1, \dots, n_1\}$ is linearly independent. Moreover, since each y'_{i_r} is a linear combination of the elements of the set $\{y_i : i = 1, \dots, n\}$, all the operators $x_{i_r} \otimes y'_{i_r}$ lie in the nest algebra $\mathcal{T}(\mathcal{N})$ and, consequently, the operator TQ satisfies all the conditions of this lemma.

Now it may happen that TQ satisfies the condition of part a), in which case it follows immediately that T is a finite sum of rank one operators lying in $\mathfrak{L}$. Alternatively, if TQ satisfies the condition of part b), then

$$TQ = TQQ_1 + TQQ_1^{\perp},$$

where Q_1 is the projection in the nest defined by

$$Q_1 = \bigwedge\{P \in \mathcal{N} : \mathfrak{X}_{1P} \text{ is linearly independent}\}$$

and

$$\mathfrak{X}_{1P} = \{Px_{i_r} : r = 1, \dots, n_1\}.$$

Applying the reasoning of part b) of the proof, either TQQ_1 satisfies the conditions in part a), thus ending the proof, or is an operator of rank n_2 less than n_1 such that

$$TQQ_1 = TQQ_1Q_2 + TQQ_1Q_2^{\perp},$$

where Q_2 is a projection defined analogously to Q_1 above. Repeating this procedure a sufficient (finite) number of times, either situation a) occurs at some point or situation b) always happens. In the first hypothesis, the proof ends and in the second hypothesis, since the rank of the relevant operators strictly decreases in each step, it is eventually possible to obtain a rank one operator $TQQ_1Q_2Q_3 \dots Q_l$ lying in $\mathfrak{L}$, thus concluding the proof.

□

Theorem 3.5. *Let $\mathcal{T}(\mathcal{N})$ be a continuous nest algebra, let $\mathfrak{L}$ be a norm closed Lie ideal of $\mathcal{T}(\mathcal{N})$ and let T be a finite rank operator in $\mathfrak{L}$. Then T can be written as a finite sum of rank one operators lying in $\mathfrak{L}$.*

Proof. Clearly, the assertion holds if $T = 0$ or if T is a rank one operator. Let T be an operator of rank $n \geq 2$ lying in the Lie Ideal $\mathfrak{L}$. It is possible to write the operator T as

$$T = \sum_{i=1}^{n} x_i \otimes y_i,$$

where, for all $i = 1, \dots, n$, the rank one operator $x_i \otimes y_i$ lies in $\mathcal{T}(\mathcal{N})$ (cf. [2, 6]). Suppose, without loss of generality, that, for all indices $i, j = 1, \dots, n$,

$$i \leq j \ \Rightarrow \ \hat{P}_{x_i} \leq \hat{P}_{x_j}.$$

This proof is divided into part a) and part b) according to $\hat{P}_{x_1} < \hat{P}_{x_2}$ or $\hat{P}_{x_1} = \hat{P}_{x_2}$, respectively.

a) Suppose that $\hat{P}_{x_1} < \hat{P}_{x_2}$. Either, for all indices $i, j = 1, \dots, n$,

$$i < j \ \Rightarrow \ \hat{P}_{x_i} < \hat{P}_{x_j},$$

and the result immediately follows from Lemma 3.3, or there exists a positive integer $k < n$ such that

$$\hat{P}_{x_1} < \cdots < \hat{P}_{x_k} = \hat{P}_{x_{k+1}}.$$

If this is the case, then the operator

$$\begin{aligned} T_{k-1} &= T - [\hat{P}_{x_k}, T] \\ &= T - \hat{P}_{x_k} T \hat{P}_{x_k}^{\perp} \\ &= T - T\hat{P}_{x_k}^{\perp} \end{aligned}$$

lies in $\mathfrak{L}$. Since, for all indices $i, j = 1, \dots, k-1$,

$$i \leq j \;\Rightarrow\; \hat{P}_{x_i} < \hat{P}_{x_j}.$$

the operator

$$T_{k-1} = \sum_{i=1}^{k-1} (\hat{P}_{x_k} x_i) \otimes y_i,$$

is a rank $k-1$ operator, and by Lemma 3.3, it follows that, for all $i = 1, \dots, k-1$, the operator $(\hat{P}_{x_k} x_i) \otimes y_i$ lies in $\mathfrak{L}$. Hence, by Theorem 3.2, all the operators $x_1 \otimes y_1, \dots, x_{k-1} \otimes y_{k-1}$ lie in the Lie ideal $\mathfrak{L}$ and, therefore,

$$S = x_k \otimes y_k + x_{k+1} \otimes y_{k+1} + \cdots + x_n \otimes y_n$$

is also an element of the Lie ideal $\mathfrak{L}$.

b) Suppose now that $\hat{P}_{x_1} = \hat{P}_{x_2}$. Either

$$\hat{P}_{x_1} = \hat{P}_{x_2} = \cdots = \hat{P}_{x_n}$$

and the result immediately follows from Lemma 3.4, or there exists a positive integer $m < n$ such that

$$\hat{P}_{x_1} = \hat{P}_{x_2} = \cdots = \hat{P}_{x_m} < \hat{P}_{x_{m+1}}.$$

Hence, the operator

$$\begin{aligned} T_m &= T - [\hat{P}_{x_{m+1}}, T] \\ &= T - \hat{P}_{x_{m+1}} T \hat{P}_{x_{m+1}}^{\perp} \\ &= T - T\hat{P}_{x_{m+1}}^{\perp} \\ &= \sum_{i=1}^{m} (\hat{P}_{x_{m+1}} x_i) \otimes y_i, \end{aligned}$$

lies in $\mathfrak{L}$. If the rank of T_m equals m, by Lemma 3.4, the operator T_m can be written as a finite sum of rank one operators lying in $\mathfrak{L}$. By Theorem 3.2 and the proof of Lemma 3.4, for all $i = 1, 2, \dots, m$, the operator $(\hat{P}_{x_{m+1}}^{\perp} x_i) \otimes y_i$ lies in $\mathfrak{L}$. In fact, the operator T_m either satisfies condition a) or condition b) in the proof of Lemma 3.4. In the first case, it was shown that, for all $i = 1, 2, \dots, m$, the operator $(\hat{P}_{x_{m+1}} x_i) \otimes y_i$ lies

in $\mathfrak{L}$ and, consequently, by Theorem 3.2, also the operator $(\hat{P}^{\perp}_{x_{m+1}} x_i) \otimes y_i$ lies in $\mathfrak{L}$.

If, on the other hand, the operator T_m satisfies condition b), the proof of Lemma 3.4 shows that there exists a projection Q in the nest with

$$\hat{P}_{x_m} < Q < \hat{P}_{x_{m+1}}, \qquad \hat{P}_{Q^{\perp} x_i} < \hat{P}_{x_{m+1}}$$

and such that, for all $i = 1, 2, \ldots, m$, the rank one operator $(Q^{\perp} x_i) \otimes y_i$ lies in $\mathfrak{L}$. Theorem 3.2 now guarantees that, for all $i = 1, 2, \ldots, m$, the operator $(\hat{P}^{\perp}_{x_{m+1}} x_i) \otimes y_i$ lies in $\mathfrak{L}$. Observe that a similar reasoning to that used to show that Q cannot be equal to I can also be used to see that Q is different from $\hat{P}_{x_{m+1}}$.

It is now possible to conclude that the equality

$$T = T_m + \sum_{i=1}^{m} (\hat{P}^{\perp}_{x_{m+1}} x_i) \otimes y_i + \sum_{i=m+1}^{n} x_i \otimes y_i, \tag{1}$$

yields that the operator S defined by

$$S = \sum_{i=m+1}^{n} x_i \otimes y_i$$

lies in the Lie ideal $\mathfrak{L}$.

In the case of the rank of T_m being less than m, the set $\{\hat{P}_{x_{m+1}} x_i : i = 1, \ldots, m\}$ is linearly dependent. There exists, nevertheless, a subset J of $\{i = 1, \ldots, m\}$ such that $\{\hat{P}_{x_{m+1}} x_i : i \in J\}$ is a maximal linearly independent subset of $\{\hat{P}_{x_{m+1}} x_i : i = 1, \ldots, m\}$ and therefore, for all $k \in J_d$, where

$$J_d = \{1, \ldots, m\} \backslash J,$$

the identity

$$\hat{P}_{x_{m+1}} x_k = \sum_{j \in J} a_{kj} \hat{P}_{x_{m+1}} x_j$$

holds. Consequently, the operator T_m can be re-written as

$$
\begin{aligned}
T_m &= \sum_{i\in J}(\hat{P}_{x_{m+1}}x_i)\otimes y_i + \sum_{k\in J_d}(\hat{P}_{x_{m+1}}x_k)\otimes y_k \\
&= \sum_{i\in J}(\hat{P}_{x_{m+1}}x_i)\otimes y_i + \sum_{k\in J_d}\Big(\sum_{j\in J}(a_{kj}\hat{P}_{x_{m+1}}x_j)\Big)\otimes y_k \\
&= \sum_{i\in J}(\hat{P}_{x_{m+1}}x_i)\otimes y_i + \sum_{j\in J}\Big(\sum_{k\in J_d}(a_{kj}\hat{P}_{x_{m+1}}x_j)\otimes y_k\Big) \\
&= \sum_{i\in J}(\hat{P}_{x_{m+1}}x_i)\otimes y_i + \sum_{j\in J}\Big(\sum_{k\in J_d}(\hat{P}_{x_{m+1}}x_j)\otimes \bar{a}_{kj}y_k\Big) \\
&= \sum_{i\in J}(\hat{P}_{x_{m+1}}x_i)\otimes y_i + \sum_{j\in J}\Big((\hat{P}_{x_{m+1}}x_j)\otimes \sum_{k\in J_d}\bar{a}_{kj}y_k\Big) \\
&= \sum_{i\in J}(\hat{P}_{x_{m+1}}x_i)\otimes \Big(y_i + \sum_{k\in J_d}\bar{a}_{ki}y_k\Big) = \sum_{i\in J}(\hat{P}_{x_{m+1}}x_i)\otimes y_i',
\end{aligned}
$$

where the sets $\{\hat{P}_{x_{m+1}}x_i : i\in J\}$ and $\{y_i' : i\in J\}$ are linearly independent. By Lemma 3.4, the operator T_m can be written as a finite sum of rank one operators lying in $\mathfrak{L}$. Observe also that, by Theorem 3.2 and the proof of Lemma 3.4, for all $i\in J$, the operator $(\hat{P}^{\perp}_{x_{m+1}}x_i)\otimes y_i'$ lies in $\mathfrak{L}$.

It follows from the equality (1) that the operator

$$
S_m = \sum_{i=1}^{m}(\hat{P}^{\perp}_{x_{m+1}}x_i)\otimes y_i + \sum_{i=m+1}^{n} x_i\otimes y_i.
$$

lies in the ideal.

It is now necessary to show that S_m can be written as a finite sum of rank one operators in $\mathfrak{L}$. Since

$$
\begin{aligned}
S_m &= \sum_{i\in J}(\hat{P}^{\perp}_{x_{m+1}}x_i)\otimes y_i + \sum_{i\in J_d}(\hat{P}^{\perp}_{x_{m+1}}x_i)\otimes y_i + \sum_{i=m+1}^{n} x_i\otimes y_i \\
&= \sum_{i\in J}(\hat{P}^{\perp}_{x_{m+1}}x_i)\otimes \Big(y_i + \sum_{k\in J_d}\bar{a}_{ki}y_k\Big) - \sum_{i\in J}(\hat{P}^{\perp}_{x_{m+1}}x_i)\otimes \Big(\sum_{k\in J_d}\bar{a}_{ki}y_k\Big) + \\
&\qquad\qquad + \sum_{i\in J_d}(\hat{P}^{\perp}_{x_{m+1}}x_i)\otimes y_i + \sum_{i=m+1}^{n} x_i\otimes y_i \\
&= \sum_{i\in J}(\hat{P}^{\perp}_{x_{m+1}}x_i)\otimes y_i' - \sum_{k\in J_d}\Big(\sum_{i\in J}(a_{ki}\hat{P}^{\perp}_{x_{m+1}}x_i)\otimes y_k\Big) + \\
&\qquad\qquad + \sum_{i\in J_d}(\hat{P}^{\perp}_{x_{m+1}}x_i)\otimes y_i + \sum_{i=m+1}^{n} x_i\otimes y_i \\
&= \sum_{i\in J}(\hat{P}^{\perp}_{x_{m+1}}x_i)\otimes y_i' + \sum_{k\in J_d}\Big(\sum_{i\in J}\Big(\hat{P}^{\perp}_{x_{m+1}}x_k - a_{ki}\hat{P}^{\perp}_{x_{m+1}}x_i\Big)\Big)\otimes y_k + \\
&\qquad\qquad + \sum_{i=m+1}^{n} x_i\otimes y_i,
\end{aligned}
$$

where the first summand is a sum of rank one operators in $\mathfrak{L}$, it follows that the operator

$$
S = \sum_{k\in J_d}\Big(\sum_{i\in J}\Big(\hat{P}^{\perp}_{x_{m+1}}x_k - a_{ki}\hat{P}^{\perp}_{x_{m+1}}x_i\Big)\Big)\otimes y_k + \sum_{i=m+1}^{n} x_i\otimes y_i
$$

lies in $\mathfrak{L}$ and has rank l less than n. The operator S can be re-written as a sum of l rank one operators lying in $\mathcal{T}(\mathcal{N})$ (cf. [2, 6]).

Applying again the same reasoning to either of the operators S of part a) or part b), according to the case, it is possible to conclude, after a finite number of steps, that T can be written as a finite sum of rank one operators in $\mathfrak{L}$. $\square$

4. A characterisation theorem

Recall that a mapping $\varphi : \mathcal{N} \to \mathcal{N}$, defined on a nest $\mathcal{N}$, is called a *homomorphism* if, for all projections P and Q in $\mathcal{N}$,

$$
P \le Q \quad \Longrightarrow \quad \varphi(P) \le \varphi(Q).
$$

A homomorphism φ is said to be *left order continuous* if, for all subsets $\mathcal{M}$ of the nest $\mathcal{N}$, the projection $\varphi(\bigvee \mathcal{M})$ is equal to the supremum $\bigvee \varphi(\mathcal{M})$.

Proposition 4.1. *Let $\mathcal{T}(\mathcal{N})$ be a continuous nest algebra associated to a nest $\mathcal{N}$ and let $\mathfrak{L}$ be a norm closed Lie ideal of $\mathcal{T}(\mathcal{N})$. Then, the mapping*

$P \mapsto P'$ defined, for all projections P in the nest $\mathcal{N}$, by

$$P' = \bigvee \left\{ P_y \in \mathcal{N} : x \otimes y \in \mathfrak{L} \ \wedge \ \hat{P}_x < P \right\} \tag{2}$$

is a left order continuous homomorphism on $\mathcal{N}$.

Proof. Let P and Q be projections in the nest $\mathcal{N}$ such that $P < Q$. It will be shown that the projection P' is less than or equal to the projection Q'.

If the projection Q' were less than the projection P', then there would exist an operator $x \otimes y$ in $\mathfrak{L}$ such that

$$\hat{P}_x < P\,, \qquad Q' < P_y.$$

Therefore, by Theorem 3.2, all operators $w \otimes z$, such that $\hat{P}_x \leq \hat{P}_w$ and $P_z \leq P_y$, would lie in $\mathfrak{L}$. Hence,

$$Q' < \bigvee \left\{ P_z \in \mathcal{N} : w \otimes z \in \mathfrak{L},\ \hat{P}_w < Q \right\}\ ,$$

yielding a contradiction, and thus concluding the proof that the mapping is a homomorphism.

To show that the order homomorphism $P \mapsto P'$ is left order continuous on $\mathcal{N}$, let $\mathcal{M}$ be a subset of the nest $\mathcal{N}$, let $\mathcal{M}'$ be the set defined by

$$\mathcal{M}' = \left\{ P' \in \mathcal{N} : P \in \mathcal{M} \right\}$$

and let the projection Q be the supremum of $\mathcal{M}$.

If the projection Q lies in $\mathcal{M}$, then, since the mapping $P \mapsto P'$ is an order homomorphism, the projection Q' coincides with the supremum $\bigvee \mathcal{M}'$.

Suppose now that the supremum Q does not lie in $\mathcal{M}$. It follows that

$$\begin{aligned} Q' &= \bigvee \left\{ P_y \in \mathcal{N} : x \otimes y \in \mathfrak{L} \ \wedge \ \hat{P}_x < Q \right\} \\ &= \bigvee \Bigg(\bigcup_{P \in \mathcal{N}, P < Q} \left\{ P_y \in \mathcal{N} : x \otimes y \in \mathfrak{L},\ \hat{P}_x < P \right\} \Bigg). \end{aligned}$$

By Theorem 3.2 and the fact that the mapping $P \mapsto P'$ is an order homomorphism,

$$\begin{aligned} Q' &= \bigvee \Bigg(\bigcup_{P \in \mathcal{M}} \left\{ P_y \in \mathcal{N} : x \otimes y \in \mathfrak{L},\ \hat{P}_x < P \right\} \Bigg) \\ &= \bigvee \{ P' \in \mathcal{N} : P \in \mathcal{M} \} \\ &= \bigvee \mathcal{M}'\ , \end{aligned}$$

which concludes the proof. □

Lemma 4.2. *Let $\mathcal{T}(\mathcal{N})$ be a continuous nest algebra associated to a nest $\mathcal{N}$, let $\mathfrak{L}$ be a norm closed Lie ideal of $\mathcal{T}(\mathcal{N})$ and let the mapping $P \mapsto P'$ be the left order continuous homomorphism* (2)*. Then, a rank one operator $x \otimes y$ lies in $\mathfrak{L}$ if and only if, for all projections P in the nest $\mathcal{N}$,*

$$P'^{\perp}(x \otimes y)P = 0.$$

It should be observed that any rank one operator satisfying the above condition must lie in the nest algebra $\mathcal{T}(\mathcal{N})$.

Proof. If $x \otimes y$ is a rank one operator in $\mathfrak{L}$ then, for all projections P in the nest,

$$P'^{\perp}(x \otimes y)P = (Px) \otimes P'^{\perp}y$$

coincides with zero. In fact, either $P \leq \hat{P}_x$, which leads to Px being equal to zero, or $\hat{P}_x < P$, which implies, by the definition of the order homomorphism $P \mapsto P'$, that $P_y \leq P'$ and therefore $P'^{\perp}y$ coincides with zero.

Conversely, let $x \otimes y$ be such that, for all projections P in $\mathcal{N}$, the operator $P'^{\perp}(x \otimes y)P$ coincides with zero. Observe that, for any projection P greater than $\hat{P}_x$, the element $P'^{\perp}y$ must be equal to zero, i.e., $P_y \leq P'$.

Notice also that a rank one operator $w \otimes z$ such that $P \leq \hat{P}_w$ and $P_z \leq P'$ must lie in the norm closed ideal $\mathfrak{L}$. Clearly, by Theorem 3.2 and the definition of the mapping $P \mapsto P'$, all operators $w \otimes z$, with $P \leq \hat{P}_w$ and $P_z < P'$, lie in $\mathfrak{L}$. If $w \otimes z$ is such that $P \leq \hat{P}_w$ and $P_z = P'$, then there exists a sequence (z_n) in

$$\bigcup_{Q \in \mathcal{N}, Q < P'} Q(\mathfrak{H}),$$

converging to z in the norm topology. Hence $(w \otimes z_n)$ is a sequence in $\mathfrak{L}$ whose limit $w \otimes z$ also lies in $\mathfrak{L}$.

It is possible to find a sequence $(x_n \otimes y)$ converging to $x \otimes y$ and such that, for all n, the projection $\hat{P}_x$ is less than the projection $\hat{P}_{x_n}$. Since, by the reasoning above, this sequence lies in $\mathfrak{L}$, it follows that $x \otimes y$ is an element of the ideal. □

Theorem 4.3. *Let $\mathcal{T}(\mathcal{N})$ be a continuous nest algebra associated to a nest $\mathcal{N}$, let $\mathfrak{L}$ be a norm closed Lie ideal of $\mathcal{T}(\mathcal{N})$ and let the mapping $P \mapsto P'$ be the left order homomorphism* (2)*. Then a finite rank operator T in the nest algebra $\mathcal{T}(\mathcal{N})$ lies in $\mathfrak{L}$ if and only if, for all projections P in $\mathcal{N}$,*

$$P'^{\perp}TP = 0.$$

Proof. The assertion trivially holds for $T = 0$. It will be assumed from now on that T is an operator of rank $n \geq 1$ in the nest algebra $\mathcal{T}(\mathcal{N})$.

If the operator T lies in $\mathfrak{L}$, then by Theorem 3.5, it can be written as a finite sum of rank one operators in the ideal. By Lemma 4.2, each of this rank one operators satisfies the condition associated to the homomorphism $P \mapsto P'$ and, therefore, also T satisfies the condition.

Conversely, suppose that T is an operator in the nest algebra with rank $n \geq 1$ and such that, for all P in $\mathcal{N}$,

$$P'^{\perp}TP = 0.$$

As a consequence, the operator T lies in the norm closed (associative) ideal

$$\mathfrak{B} = \{S \in \mathcal{T}(\mathcal{N}) : P'^{\perp}SP = 0\}$$

of the nest algebra $\mathcal{T}(\mathcal{N})$ and thus, by Theorem 3.5, there exist finitely many operators $x_i \otimes y_i$ in $\mathfrak{B}$ such that

$$T = \sum_i x_i \otimes y_i.$$

Therefore, by Lemma 4.2, for all i, the operator $x_i \otimes y_i$ lies in $\mathfrak{L}$ and thus also T is an operator in this ideal. □

Acknowledgments. The author wishes to thank I. G. Todorov for his helpful remarks during the preparation of the manuscript. The financial support of CEAF, through a FCT grant under the Research Units Pluriannual Funding Programme, is also gratefully acknowledged.

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